changes between the proposal stage and actual deployment, the inclusion of propellant mass in the equation derived for spacecraft wet mass allows changes to the design lifetime and design altitude to be explicitly taken into account.

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## References

[1] de Weck, O. L., and Chang, D. D., "Architecture Trade Methodology for LEO Personal Communication Systems," AIAA Paper 2002-1866, May 2002.

[2] Pritchard, W. L., "Estimating the Mass and Power of Communications Satellites," *International Journal of Satellite Communications*, Vol. 2, No. 2, 1984, pp. 107–112.

[3] Richharia, M. *Satellite Communications Systems: Design Principles*, McGraw–Hill, New York, 1995, pp. 300–306.

[4] Larson, W. J., and Wertz, J. R. (eds.), *Space Mission Analysis and Design*, 3rd ed., Microcosm Press, El Segundo, CA, and Kluwer Academic, Boston, 2001, Chap. 10.

[5] Saleh, J. H., Hastings, D. E., and Newman, D. J., "Spacecraft Design Lifetime," *Journal of Spacecraft and Rockets*, Vol. 39, No. 2, 2002, pp. 244–257.

[6] Khuri, A. I., and Cornell, J. A., *Response Surfaces: Designs and Analyses*, Marcel Dekker, New York, 1996.

[7] Lutz, E., Werner, M., and Jahn, A., *Satellite Systems for Personal and Broadband Communications*, Springer-Verlag, New York, 2000, pp. 375–378, 389–395.

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# Analysis of Optimal Weaving Frequency of Maneuvering Targets

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## Nomenclature

$a_c$ = guidance command, m[2]/s
$a_L$ = missile acceleration, m[2]/s
$a_T$ = target acceleration, m[2]/s
$N$ = effective navigation ratio
$t_{go}$ = time to go, s
$V_{cl}$ = closing velocity, m/s
$y$ = relative separation between a missile and target, m
$\zeta$ = flight-control system damping
$\lambda$ = line-of-sight angle, rad
$\tau$ = flight-control system time constant, s
$\omega$ = flight-control system natural frequency, rad/s
$\omega_z$ = airframe zero frequency, rad/s

## Introduction

MANEUVERS present the best strategy for missiles to achieve their goals. Sinusoidal or weave maneuvers of a target can make it difficult for a pursuing missile to obtain an intercept. Optimal control and game theory have been used in attempts to formulate precisely and solve the problems of optimal pursuit and evasion.[1–4] Unfortunately, it is difficult to build a realistic analytical model that can be utilized in practice. Deterministic optimal control laws require full information about a target and missile flight parameters. Such complete information is never available. However, it is reasonable to assume that the evader (target) knows the types of missiles and guidance laws that can be used for interception, but does not know the time to go until intercept of the pursuing missile. In this case, a properly chosen periodic maneuver sequence offers the target its best chance of survival.[5,6]

Miss distances depend on the frequency of maneuvers. It is shown that their functional relationship has a maximum, that is, there exists a frequency for which the amplitude of the miss distance has a maximum. The procedure of determination of the optimal frequency is presented.

This Note considers the influence of the target weave maneuver on a third-order proportional navigation guidance system, presented in the form that is commonly used in practice. The analogous analysis for a simplified first-order proportional navigation guidance system is given in Ref. 6. The generalized analysis in this Note makes it possible to utilize in practice the relations obtained. A closed-form solution for the miss distance as a function of the effective navigation ratio, guidance system time constant, natural frequency, and damping and weave maneuver amplitude and frequency is derived. The procedure of determination of the optimal frequency, that is, the frequency that maximizes the steady-state miss distance amplitude, is described.

## Weave Maneuver Analysis

A block diagram of the guidance system under consideration is given in Fig. 1. It has the same structure as that in Refs. 4–6. Here, missile acceleration $n_L$ is subtracted from target acceleration $n_T$, and the result is integrated to obtain relative position $y$, which, at the end of flight $t_f$, is the miss distance $y(t_f)$. A division by range (closing velocity $V_{cl}$ multiplied by time to go $t_{go}$ until intercept) yields the geometric line-of-sight (LOS) angle $\lambda$, where the time to go is defined as $t_{go} = t_f - t$. It is assumed that the missile seeker, represented as a perfect differentiator, effectively provides a measurement of the rotation rate of LOS from the interceptor to the target. The filter dynamics are neglected. (For the problem under consideration, the time lag of a filter can be taken into account by an increase in the time constant of the flight-control system.) Perfect estimation of the LOS rate is assumed to generate a guidance command $n_c$ based on the proportional navigation law.

The flight-control system guides the missile to follow this acceleration command. The flight-control system dynamics, which combines its airframe and autopilot dynamics, are represented by

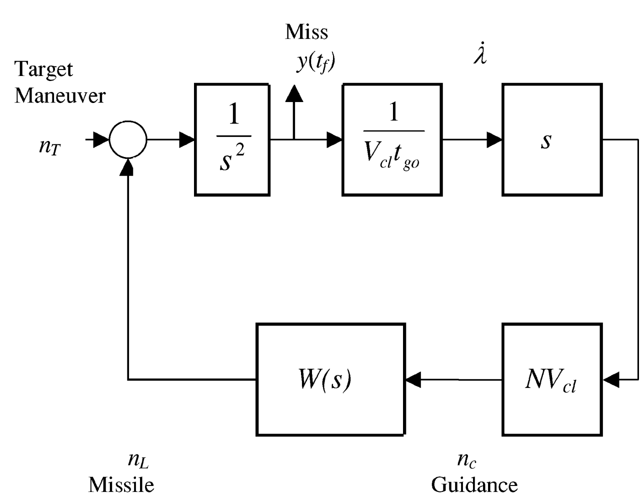


**Fig. 1 Missile guidance model.**

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the following transfer function:

$$W(s) = \frac{1 - s^2/\omega_z^2}{(1+\tau s)\left[1 + (2\zeta/\omega_M)s + s^2/\omega_M^2\right]} \tag{1}$$

with the flight-control system damping $\zeta$, natural frequency $\omega_M$, time constant $\tau$, and the airframe zero frequency $\omega_z$.

According to Ref. 7, in the complex domain, the miss distance at time $t_f$ can be presented as

$$Y(t_f, s) = \exp\left[N \int_{\infty}^{s} H(\sigma)\,\mathrm{d}\sigma\right] Y_T(s) \tag{2}$$

where $Y_T(s)$ is the Laplace transform of target distance, and

$$H(s) = W(s)/s \tag{3}$$

The miss distance due to a weaving target will be evaluated by determination of the magnitude of the steady-state component of Eq. (2) when the input is a unit harmonic signal, for example, a unit sine wave, of target acceleration with frequency $\omega$ ($n_T = 1g \sin\omega t$, where $g$ is acceleration of gravity), that is, from the expression

$$P(t_f, i\omega) = \exp\left[N \int_{\infty}^{i\omega} H(\sigma)\,\mathrm{d}\sigma\right] \frac{g}{(i\omega)^2} \tag{4}$$

where $P(t_f, i\omega)$ is the frequency response with respect to the miss distance at moment $t_f$.

The integral

$$\int_{\infty}^{i\omega} H(\sigma)\,\mathrm{d}\sigma$$

can be calculated when $H(s)$ is written in the form

$$H(s) = \frac{A}{s} + \frac{B/\tau}{s + 1/\tau} + \frac{Cs + D}{s^2/\omega_M^2 + (2\zeta/\omega_M)s + 1} \tag{5}$$

where the coefficients $A$, $B$, $C$, and $D$ can be calculated as

$$A = 1, \qquad B = \frac{\tau^2 - 1/\omega_z^2}{2\xi/\omega_M - \tau - 1/\tau\omega_M^2}$$

$$C = -\frac{1}{\omega_M^2} - \frac{B}{\tau\omega_M^2}, \qquad D = -\frac{2\xi}{\omega_M} + C\tau\omega_M^2 \tag{6}$$

Integrals of the first two terms equal, respectively, $\ln(i\omega) = \ln|\omega| + i\arg(i\omega)$ and $(B/\tau)\ln(i\omega + 1/\tau) = (B/2\tau)\ln(\omega^2 + 1/\tau^2) + i(B/\tau)\arg(i\omega + 1/\tau)$. We do not specify the arguments of the preceding expressions here because they are not used in determining the amplitude characteristic $|P(t_f, i\omega)|$ of $P(t_f, i\omega)$. We also do not consider the lower limit of integration because, as shown in Ref. 7, for the given structure of $H(s)$ and $W(s)$, the result of evaluation of the whole expression at the lower infinite limit of integration equals zero.

The integral of the third term of Eq. (5) is calculated by separation of the real and imaginary parts of the integrand. We will present only the real part of integration because only this part is needed to obtain $|P(t_f, i\omega)|$ (see Appendix):

$$\frac{C}{4}\omega_M^2 \ln\left[\left(\omega_M^2 - \omega^2\right)^2 + 4\omega_M^2\zeta^2\right] + \frac{\omega_M(D - \zeta\omega_M C)}{2\sqrt{1-\zeta^2}}$$
$$\times\left(\tan^{-1}\frac{\omega - \omega_M\sqrt{1-\zeta^2}}{\zeta\omega_M} - \tan^{-1}\frac{\omega + \omega_M\sqrt{1-\zeta^2}}{\zeta\omega_M}\right)$$

By substitution of the real components of the integrals just calculated into Eq. (4), we obtain the amplitude characteristic $|P(t_f, i\omega)|$ in the following form:

$$|P(t_f, \omega)| = \omega^N (\omega^2 + 1/\tau^2)^{BN/2\tau}$$
$$\times\left[\left(\omega_M^2 - \omega^2\right)^2 + 4\omega_M^2\zeta^2\right]^{CN\omega_M^2/4} \exp(\,)(g/\omega^2) \tag{7}$$

where

$$\exp(\,) = \exp\left[N\frac{\omega_M(D - \zeta\omega_M C)}{2\sqrt{1-\zeta^2}}\right.$$
$$\left.\times\left(\tan^{-1}\frac{\omega - \omega_M\sqrt{1-\zeta^2}}{\zeta\omega_M} - \tan^{-1}\frac{\omega + \omega_M\sqrt{1-\zeta^2}}{\zeta\omega_M}\right)\right] \tag{8}$$

Analysis of Eq. (7) shows that the amplitude characteristic has a maximum with respect to the frequency $\omega$. It can be determined easily by simulation of relationship (7).

Figure 2 shows this relationship for $N = 3$, $\omega_M = 20$ rad/s, $\omega_z = 5$ rad/s, $\tau = 0.5$ s, and $\zeta = 0.7$. From Fig. 2, we can expect the maximum miss distance (peak miss) for a target maneuver with a frequency of 1.4 rad/s, that is, with a period of 4.48 s. Figure 3 presents the simulation result for the linearized two-dimensional model in Fig. 1 and a target acceleration $n_T = 1g \sin 1.4t$, which corresponds to the optimal maneuver frequency determined earlier. The peak miss amplitude in Fig. 3 for this frequency is in full correspondence with the peak miss in Fig. 2 obtained from Eq. (7). The

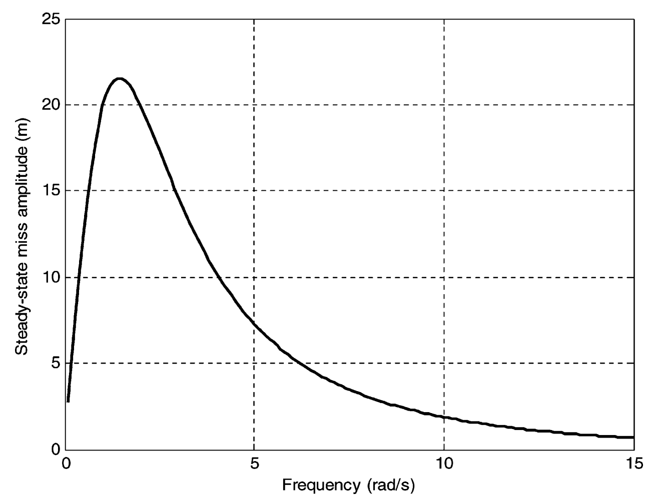


**Fig. 2 Amplitude characteristics of frequency response.**

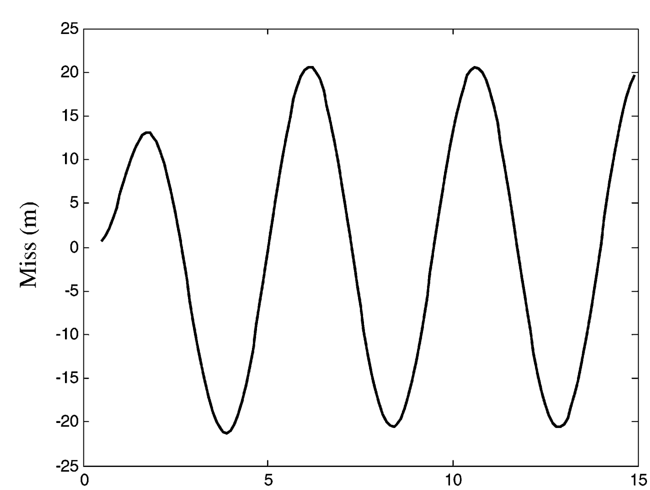


**Fig. 3 Optimal weaving frequency and peak miss distance based on analytical analysis agree with simulation results.**

**Table 1 Influence of flight-control system parameters on optimal weaving frequency and peak miss distance**

| Case number | $\omega_z$, rad/s | $\tau$, s | $\zeta$ | $\omega_M$, rad/s | Peak miss, m | $\omega_{\mathrm{opt}}$, rad/s |
|---|---|---|---|---|---|---|
| 1 | 5 | 0.5 | 0.7 | 20 | 21.54 | 1.4 |
| 2 | 10 | 0.5 | 0.7 | 20 | 4.25 | 1.5 |
| 3 | 20 | 0.5 | 0.7 | 20 | 2.86 | 1.3 |
| 4 | 100 | 0.5 | 0.7 | 20 | 2.5 | 1.3 |
| 5 | 5 | 0.2 | 0.7 | 20 | 56.6 | 8 |
| 6 | 5 | 0.6 | 0.7 | 20 | 21.19 | 1.2 |
| 7 | 5 | 0.7 | 0.7 | 20 | 21.5 | 1 |
| 8 | 5 | 0.5 | 0.6 | 20 | 12 | 1.5 |
| 9 | 5 | 0.5 | 0.8 | 20 | 39.72 | 1.4 |
| 10 | 5 | 0.5 | 0.7 | 10 | 9.25 | 1.3 |
| 11 | 5 | 0.5 | 0.7 | 30 | 70.4 | 1.5 |

analysis of the optimal weaving frequency given earlier can provide the basis for a more rigorous analysis later by the use of a more detailed engagement model.

The flight-control system of a tail-controlled endoatmospheric missile with the indicated parameters was considered in Ref. 8. It was shown that, at high altitude, the performance of a tail-controlled aerodynamic missile can deteriorate because of the existence of low-frequency right half-plane zeroes $\omega_z$.

Table 1 shows the influence of the flight-control system parameters on the miss amplitude and the optimal weaving frequency $\omega_{\mathrm{opt}}$; deviations were considered with respect to the values utilized in Figs. 2 and 3. As seen from Table 1, the peak miss decreases drastically when $\omega_z \geq 10$ rad/s, but right half-plane zeroes do not significantly influence the optimal weaving frequency. The increase of the amplitude miss for smaller values of time constant and larger values of damping and natural frequency is stipulated by the unsatisfactory dynamic properties of the flight-control system (peak overshoot, settling time, etc.).

## Conclusions

The procedure of determination of the optimal frequency for which the amplitude of the miss distance has a maximum is presented. The optimal frequency is evaluated approximately by calculation of the steady-state component of the linearized missile guidance model based on the unit sinusoidal target signal. The established existence of a maximizing frequency offers an approach for development of an optimal evasive maneuver design. The optimization procedure described also provides a way to evaluate the worst-case scenario when missiles engaging maneuvering targets are developed.

## Appendix: Integral Calculation

The integral from the third term of Eq. (5) can be presented as

$$\int_\infty^s \frac{Cs + D}{s^2/\omega_M^2 + (2\zeta/\omega_M)s + 1}\,\mathrm{d}s = \int_\infty^s \frac{C\omega_M^2 s + D\omega_M^2}{s^2 + 2\omega_M\zeta s + \omega_M^2}\,\mathrm{d}s$$

$$= \frac{C\omega_M^2}{2}\,\ln\left(s^2 + 2\omega_M\zeta s + \omega_M^2\right) - \int_\infty^s \frac{\omega_M^2(\zeta\omega_M C - D)}{s^2 + 2\omega_M\zeta s + \omega_M^2}\,\mathrm{d}s$$

$$= \frac{C\omega_M^2}{2}\,\ln\left(s^2 + 2\omega_M\zeta s + \omega_M^2\right) + \omega_M^2(D - \zeta\omega_M C)$$

$$\times \frac{1}{\omega_M\sqrt{1-\zeta^2}}\arctan\frac{s + \zeta\omega_M}{\omega_M\sqrt{1-\zeta^2}}$$

$$= \frac{C\omega_M^2}{2}\,\ln\left(s^2 + 2\omega_M\zeta s + \omega_M^2\right) + \omega_M^2(D - \zeta\omega_M C)$$

$$\times \frac{1}{\omega_M\sqrt{1-\zeta^2}}\frac{1}{2i}\,\ln\frac{i\omega_M\sqrt{1-\zeta^2} - (s + \zeta\omega_M)}{i\omega_M\sqrt{1-\zeta^2} + (s + \zeta\omega_M)}$$

For $s = i\omega$, the last expression can be written as

$$\frac{C}{2}\omega_M^2\,\ln\left(\omega_M^2 - \omega^2 + 2j\omega_M\zeta\omega\right)$$

$$- i\,\frac{\omega_M(D - \zeta\omega_M C)}{2\sqrt{1-\zeta^2}}\,\ln\frac{i\left(-\omega + \omega_M\sqrt{1-\zeta^2}\right) - \zeta\omega_M}{i\left(\omega + \omega_M\sqrt{1-\zeta^2}\right) + \zeta\omega_M}$$

$$= \frac{C}{4}\omega_M^2\,\ln\left(\left(\omega_M^2 - \omega^2\right)^2 + 4\omega_M^2\zeta^2\right)$$

$$+ \frac{\omega_M(D - \zeta\omega_M C)}{2\sqrt{1-\zeta^2}}\left(\tan^{-1}\frac{\omega - \omega_M\sqrt{1-\zeta^2}}{\zeta\omega_M}\right.$$

$$\left. - \tan^{-1}\frac{\omega + \omega_M\sqrt{1-\zeta^2}}{\zeta\omega_M}\right) + i\,\mathrm{Im}(\,)$$

Here the symbol acrtan is used to denote the inverse tangent function of the complex variable, and the symbol $\tan^{-1}$ denotes the inverse tangent function of the real variable that characterizes the argument of the complex variable of the logarithmic function.

## References


[1]Ben-Asher, J. Z., and Yaesh, I., *Advances in Missile Guidance Theory*, Vol. 180, Progress in Astronautics and Aeronautics, AIAA, Reston, VA, 1998.

[2]Ho, Y. C., Bryson, A. E., Jr., and Baron, S., "Differential Games Optimal Pursuit–Evasion Strategies," *IEEE Transactions on Automatic Control*, Vol. 10, No. 4, 1965, pp. 385–389.

[3]Yanushevsky, R., "On the Application of the Functional Analysis and the Theory of Games in Multivariable Control Problems," *Proceedings of the 2nd IFAC Symposium on Multivariable Technical Control Systems*, Dusseldorf, Germany, 1971.

[4]Shinar, J., and Steinburg, D., "Analysis of Optimal Evasive Maneuvers Based on a Linearized Two-Dimensional Model," *Journal of Aircraft*, Vol. 14, No. 8, 1977, pp. 795–802.

[5]Zarchan, P., *Tactical and Strategic Missile Guidance*, Vol. 176, Progress in Astronautics and Aeronautics, AIAA, Reston, VA, 1997.

[6]Zarchan, P., "Proportional Navigation and Weaving Targets," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 5, 1995, pp. 969–974.

[7]Gurfil, P., Jodorkovsky, M., and Guelman, M., "Design of Nonsaturating Guidance Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 4, 2000, pp. 693–700.

[8]Zarchan, P., Greenberg, E., and Alpert, J., "Improving the High Altitude Performance of Tail-Controlled Endoatmospheric Missiles," AIAA Paper 2002-4770, Aug. 2002.


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